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Exactly solvable models for the Schrödinger equation from generalized Darboux transformations

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Abstract. Exactly solvable models based on Darboux transformations of a generalized Schrödinger equation are studied. The formulation allows not only a unified description of the standard inverse scattering problems at fixed energy and at fixed angular momentum but also includes problems with a linear relationship between λ^2 and k^2 as well as the treatment of λ^2- or k^2-dependent potentials. Based on a characteristic symmetry property of Darboux transformations, generalized integral equations are derived and discussed with respect to the standard equations of inverse scattering theory. New exactly solvable models for the generalized Schrödinger equation are constructed.

1. Introduction

Exactly solvable quantum Hamiltonians are of great interest for the investigation of quantum scattering problems [1]. In principle they result as special solutions of the integral equations of the inverse scattering theory [2], although in many cases they are not directly derived from these equations. The usual concept of a solvable model comprises the knowledge of the whole set of eigenfunctions and eigenvalues in a closed form. There are comparatively few potentials satisfying such a criterion [3, 4].

Darboux transformations [5] represent a powerful tool in generating families of isospectral Hamiltonians. Originally a theorem on second-order differential equations, it was generalized by Crum [6] and used in one- and three-dimensional inverse scattering problems [7, 8]. Extensions of the Darboux theorem to a wide class of linear partial differential equations as well as to differential–difference and difference–difference linear evolution equations were first considered by Matveev [10] in 1979. In this context Darboux transformations, which are directly related to the Bäcklund transformations [9], became very popular as basic ingredients in the study of nonlinear partial differential equations [11].

In this paper we consider Darboux transformations in view of their application in inverse scattering problems. Darboux transformations yield exactly solvable models which are used to construct a local, spherically symmetric potential from a given S-matrix [12, 13]. In principle Darboux transformations lead only to explicit relations between the transformed Hamiltonian and a background Hamiltonian, together with their eigenfunctions. All features of the background Hamiltonian must be known, but

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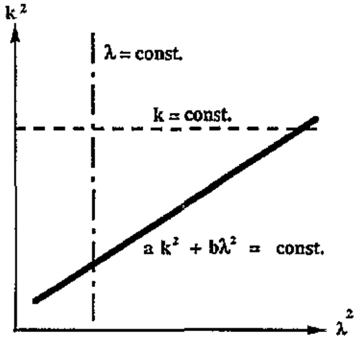

Figure 1. The (λ, k)-value at which the S-matrix is required in different inverse scattering problems. The axis for the complex quantities λ^2 and k^2 must be understood symbolically. The energy k^2 is kept constant in exactly solvable models at fixed k (dashed line), the angular momentum λ is kept constant in inverse scattering problems at fixed l (dashed dotted line). The solid line indicates general inverse scattering problems with mutual linear dependence of λ^2 and k^2.

not necessarily in closed analytic form. Therefore, Darboux transformations extend the class of potentials for inverse scattering problems far beyond the exactly solvable ones. Nevertheless we retain for those Hamiltonians the expressions *exactly solvable models*. In many cases the additional S-matrix associated with the Darboux transformation is of rational form and thus flexible enough to approximate experimentally given S- matrices.

The exactly solvable models we are dealing with allow us to establish an exact correspondence between a scattering matrix which depends on one continuous parameter and a local potential $V(r)$ depending on the continuous parameter r. On the other hand, the S-matrix depends on two continuous variables, the wavenumber k and the angular momentum $\lambda = l + 1/2$, where l is the angular momentum quantum number. This suggests that the inverse problem is overdetermined, and explains why traditionally one variable of the S-matrix is kept constant. Correspondingly one distinguishes between inverse problems at *fixed k* and at *fixed λ*.

Rudyak and Zakhariev [14] suggested Darboux transformations for a slightly more general Sturm–Liouville problem than the radial Schrödinger equation, namely

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\lambda_0^2 - \frac{1}{4}}{r^2} + k_0^2 - V(r)\right] \psi(\theta, r) = \theta^2 h(r) \psi(\theta, r) \tag{1}$$

with the potential $V(r)$ and θ, k_0, λ_0 being complex numbers. In the corresponding inverse scattering problem θ is the continuous variable mentioned above. Depending on the specific choice of $h(r)$ this equation reduces to the radial Schrödinger equation for different inverse scattering problems. For $h(r) = -1$ equation (1) represents an eigenvalue problem in the wavenumber k, which is associated with the inverse scattering problem at fixed λ. Alternatively, one can obtain the associated Schrödinger equation for the inverse scattering problem at fixed k by the choice $h(r) = 1/r^2$. Thus, a unified description of the traditional inverse scattering problems can be based on (1). Furthermore the Sturm–Liouville ansatz (1) gives rise to new inverse scattering problems [14] where λ^2 and k^2 obey a linear relationship (figure 1). Even more appealing from the point of view of applications to realistic systems is the possibility of [15] treating potentials with linear k^2- and λ^2-dependence by a simple choice of $h(r)$.

It is the aim of this paper to study exactly solvable models for general inverse scattering problems (1) via Darboux transformations. For this purpose we start from the Darboux transformations [14] of the equation (1) and formulate a compact matrix extension. In a further step we show how and under what conditions these can be cast into general integral equations of inverse scattering theory. The study of equation (1) *leads to new exactly solvable models with an explicit relationship between the* S-matrix and the potential, which can be applied in inverse scattering problems.

In section 2 we define Darboux transformations and introduce a matrix generalization. The known Bargmann schemes of Lipperheide and Fiedeldey [12], the potentials of Theis [16] at fixed λ and the extended scheme of Leeb *et al* [17] will turn out to be special examples of exactly solvable models based on these Darboux transformations. An important role in our considerations is played by a symmetry property which gives rise to new transformations. This property is also useful in constructing the kernels of associated integral equations, as discussed in section 3. As special examples of these generalized integral equations we obtain the Gel'fand–Levitan equation in the degenerate case [20] and the integral equations of the inverse scattering problem of Burdet *et al* [19]. In section 4 we describe some new exactly solvable models which arise from generalized Darboux transformations and give their basic properties. Finally, section 5 contains some concluding remarks.

2. Darboux transformations

A very fruitful tool for obtaining exact solutions for second-order differential equations was introduced by Darboux over a century ago [4]. In the present paper we consider the Darboux transformations for the Sturm–Liouville equation (1). In the following we assume that we know the whole set of solutions $\psi_0(\theta, r)$ of the differential equation (1) with the potential $V_0(r)$. Using two specific solutions $\eta_0(\gamma, r)$ and $\zeta_0(\alpha, r)$ of (1) with $\theta=\gamma$ and $\theta=\alpha$, respectively, we can define a new function

$$\eta_1(\gamma, r) := C\frac{W[\eta_0(\gamma, r); \zeta_0(\alpha, r)]}{\sqrt{h(r)}\zeta_0(\alpha, r)} \tag{2}$$

where

$$W[\eta_0(\gamma, r); \zeta_0(\alpha, r)] := \eta_0(\gamma, r)\left(\frac{\mathrm{d}}{\mathrm{d}r}\zeta_0(\alpha, r)\right) - \left(\frac{\mathrm{d}}{\mathrm{d}r}\eta_0(\gamma, r)\right)\zeta_0(\alpha, r) \tag{3}$$

is the Wronskian and C is an arbitrary constant. This is a very simple Darboux transformation of (1), and some of its features have already been discussed by Rudyak and Zakhariev [14]. After straightforward calculations the new function $\eta_1(\gamma, r)$ turns out to be a solution of the Sturm–Liouville equation (1) with the new potential $V_1(r)$, which can be written in the compact form

$$V_1(r) = V_0(r) - 2h(r)^{1/2}\frac{\mathrm{d}}{\mathrm{d}r}\left[h(r)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(h(r)^{1/4}\zeta_0(\alpha, r)\right)\right]. \tag{4}$$

It is important to note that the modification of the potential depends only on the function $\zeta_0(\alpha, r)$. Darboux transformations of two linearly independent solutions

$\eta_0(\gamma, r)$, $\xi_0(\gamma, r)$ yield two linearly independent solutions $\eta_1(\gamma, r)$, $\xi_1(\gamma, r)$ of (1) with the potential $V_1(r)$. Hence the whole manifold of solutions associated with $V_1(r)$ is known.

In order to obtain Darboux transformations of sufficient complexity to be suitable for physical scattering processes one must generalize (3). A first step is the construction of new functions $\eta_2(\gamma, r)$ from *three* known solutions $\eta_0(\gamma, r)$, $\zeta_0(\alpha, r)$ and $\xi_0(\beta, r)$. A straightforward extension of (3) yields

$$\eta_2(\gamma, r) := h(r)^{-1/4} \frac{\tilde{A}(r)}{W[\xi_0(\beta, r); \zeta_0(\alpha, r)]} \tag{5a}$$

$$\tilde{A}(r) := \tilde{C}\, h(r)^{-3/4} \det \begin{pmatrix} \eta_0(\gamma, r) & \xi_0(\beta, r) & \zeta_0(\alpha, r) \\ \eta_0'(\gamma, r) & \xi_0'(\beta, r) & \zeta_0'(\alpha, r) \\ \eta_0''(\gamma, r) & \xi_0''(\beta, r) & \zeta_0''(\alpha, r) \end{pmatrix} \tag{5b}$$

where the prime denotes the derivative with respect to r. The functions $\eta_2(\gamma, r)$ are solutions of the Sturm–Liouville equation (1) with the potential

$$V_2(r) = V_0(r) - 2h(r)^{1/2} \frac{\mathrm{d}}{\mathrm{d}r} \left(h(r)^{-1/2} \frac{\mathrm{d}}{\mathrm{d}r} \ln W[\xi_0(\beta, r); \zeta_0(\alpha, r)] \right). \tag{6}$$

The transformation (5) is equivalent to an iteration of (3) using $\tilde{C} = C^2$. For the special choice $\tilde{C} = (\gamma^2 - \beta^2)^{-1}$, the transformation (5) can be written in the form

$$\eta_2(\gamma, r) = \eta_0(\gamma, r) - \frac{\alpha^2 - \beta^2}{\gamma^2 - \beta^2} \frac{W[\eta_0(\gamma, r); \xi_0(\beta, r)]}{W[\zeta_0(\alpha, r); \xi_0(\beta, r)]} \zeta_0(\alpha, r). \tag{7}$$

This procedure can be extended further. We obtain more general Darboux transformations by defining

$$\eta_2(\gamma, r) := \eta_0(\gamma, r) - \mathscr{X}(r)^T \mathscr{Y}(r)^{-1} \zeta(r) \tag{8a}$$

with the vectors

$$\mathscr{X}(r) := \begin{pmatrix} \dfrac{W[\eta_0(\gamma, r); \xi_0(\beta_1, r)]}{\gamma^2 - \beta_1^2} \\ \dfrac{W[\eta_0(\gamma, r); \xi_0(\beta_2, r)]}{\gamma^2 - \beta_2^2} \\ \cdots \\ \dfrac{W[\eta_0(\gamma, r); \xi_0(\beta_N, r)]}{\gamma^2 - \beta_N^2} \end{pmatrix} \qquad \zeta_0(r) := \begin{pmatrix} \zeta_0(\alpha_1, r) \\ \zeta_0(\alpha_2, r) \\ \cdots \\ \zeta_0(\alpha_N, r) \end{pmatrix} \tag{8b}$$

and the N-dimensional matrix

$$\mathscr{Y}(r) := \begin{pmatrix} \dfrac{W[\zeta_0(\alpha_1, r); \xi_0(\beta_1, r)]}{\alpha_1^2 - \beta_1^2} & \text{---} & \dfrac{W[\zeta_0(\alpha_1, r); \xi_0(\beta_N, r)]}{\alpha_1^2 - \beta_N^2} \\ \text{---} & \text{---} & \text{---} \\ \dfrac{W[\zeta_0(\alpha_N, r); \xi_0(\beta_1, r)]}{\alpha_N^2 - \beta_1^2} & \text{---} & \dfrac{W[\zeta_0(\alpha_N, r); \xi_0(\beta_N, r)]}{\alpha_N^2 - \beta_N^2} \end{pmatrix} \tag{8c}$$

The function (8) is a straightforward generalization of (7) depending on the $2N$ known solutions $\zeta_0(\alpha_i, r)$, $\xi_0(\beta_i, r)$, $i=1,2,\ldots,N$ of the Sturm–Liouville equation (1) with the potential V_0. The transformed functions (8) are solutions of the Sturm–Liouville equation (1) with the potential

$$V_2(r)=V_0(r)-2h(r)^{1/2}\frac{\mathrm{d}}{\mathrm{d}r}\left(h(r)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}r}\ln\det\mathcal{Y}\right) \tag{9}$$

which has the characteristic structure of a Bargmann-type potential.

The Darboux transformations introduced in this paper are valid for any support of the Sturm–Liouville equation (1). In particular they can be applied on the full axis in one-dimensional problems as well as on the half-axis for the radial equations in partial wave expanded three-dimensional problems. Furthermore, no assumption has been made about specific boundary conditions of the solutions $\zeta_0(\alpha_i, r)$ and $\xi_0(\beta_i, r)$. If the behaviour of these solutions for $r\to\infty$ is specified, we can deduce the corresponding scattering matrix from the Darboux-transformed wavefunctions in closed form.

The exactly solvable models at fixed angular momentum introduced by Theis [16] are special cases of (8, 9) for $h(r)=-1$, choosing the functions $\zeta_0(\alpha_i, r)$, $\xi_0(\beta_i, r)$, $i=1,2,\ldots,N$ to be regular and Jost solutions, respectively. On the other hand, the choice $h(r)=1/r^2$ leads to the models at fixed energy investigated by Leeb *et al.* [17], which also use regular and Jost solutions, and reduce to the very useful rational and non-rational Bargmann schemes developed by Lipperheide and Fiedeldey [12]. Thus the general Darboux transformations of the Sturm–Liouville equation (1) allow a unified description of these models.

Concluding this presentation of the Darboux transformations we point out some interesting properties of them. First of all, there exists a useful symmetry relation. Consider the two Darboux transformations (2) and (5). They can be written in the form

$$\xi_1(\gamma, r):=h(r)^{-1/4}\frac{A(r)}{B(r)} \tag{10}$$

where A and B are functions of r, which may also depend on other parameters. The function $\xi_1(\gamma, r)$ is a solution of the Sturm–Liouville equation (1) with the potential

$$V_1(r)=V_0(r)-2h(r)^{1/2}\frac{\mathrm{d}}{\mathrm{d}r}\left(h(r)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}r}\ln B(r)\right) \tag{11}$$

if the functions A, B fulfill the condition

$$h(r)^{1/4}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(h(r)^{-1/4}A(r)B(r)\right)-4\left(\frac{\mathrm{d}}{\mathrm{d}r}A(r)\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}B(r)\right)$$
$$=\left[\gamma^2h(r)+V_0(r)+\frac{\lambda_0^2-\frac{1}{4}}{r^2}-k_0^2\right]A(r)B(r). \tag{12}$$

This can easily be verified by evaluating the Sturm–Liouville equation (1) with the potential $V_1(r)$ and the function $\xi_1(\gamma, r)$. The relation (12) is symmetric with respect to an interchange of A and B. This property implies that

$$\bar{\xi}_1(\gamma, r) := h(r)^{-1/4} \frac{B(r)}{A(r)} \tag{13}$$

is conversely a solution of the Sturm–Liouville equation (1) with the potential

$$\bar{V}_1(r) = V_0(r) - 2h(r)^{1/2} \frac{\mathrm{d}}{\mathrm{d}r}\left(h(r)^{-1/2} \frac{\mathrm{d}}{\mathrm{d}r} \ln A(r)\right). \tag{14}$$

Let us consider some particular cases. If $A(r) = 1$ it is obvious from (12) that the choice $h(r)^{-1/4} B(r) = \zeta_0(\alpha, r)$ will lead to the simplest transformation associated with the potential (4). Another important type of function is obtained applying the symmetry relation to the transformations (2). Defining $A(r) := C\, W[\xi_0(\beta, r); \zeta_0(\alpha, r)]$ and $B(r) := h(r)^{1/4} \zeta_0(\alpha, r)$ yields the function

$$\xi_2(\beta, r) := \frac{\zeta_0(\alpha, r)}{C\, W(\xi_0(\beta, r); \zeta_0(\alpha, r)]} \tag{15}$$

which is a new solution of the Sturm–Liouville equation (1) with the potential (6) of the iterated Darboux transformation (2). A generalization of (15) leading to a solution associated with the potential (9) can also be given by

$$\boldsymbol{\xi}_2(r) := \mathcal{Y}(r)^{-1} \boldsymbol{\zeta}_0(r) \tag{16a}$$

where $\mathcal{Y}(r)$ and $\boldsymbol{\zeta}_0(r)$ are given in (8) and $\boldsymbol{\xi}_2(r)$ is the vector

$$\boldsymbol{\xi}_2(r) := \begin{pmatrix} \xi_2(\beta_1, r) \\ \xi_2(\beta_2, r) \\ \cdots \\ \xi_2(\beta_N, r) \end{pmatrix}. \tag{16b}$$

Finally, we construct the *inverse transformation* to the iterated Darboux transformation (7). For this purpose we define the operator T_{20}, which transforms $\eta_0(\gamma, r)$ into $\eta_2(\gamma, r)$:

$$\eta_2(\gamma, r) =: T_{20} \eta_0(\gamma, r) = \eta_0(\gamma, r) - \frac{\alpha^2 - \beta^2}{\gamma^2 - \beta^2} \frac{W[\eta_0(\gamma, r); \xi_0(\beta, r)]}{W[\zeta_0(\alpha, r); \xi_0(\beta, r)]} \zeta_0(\alpha, r). \tag{7a}$$

As shown above, the symmetry property of Darboux transformations (12) leads to new solutions (15) associated with the same potential V_2. In analogy to (15) we can define the function

$$\zeta_2(\alpha, r) := \frac{\xi_0(\beta, r)}{C'\, W[\xi_0(\beta, r); \zeta_0(\alpha, r)]} \tag{15a}$$

which is a solution for $\theta = \alpha$ of (1) with the potential $V_2(r)$. The two solutions $\xi_2(\beta, r)$ and $\zeta_2(\alpha, r)$, together with the solutions $\eta_2(\gamma, r)$ are used to formulate the inverse of the transformation (7*a*), namely,

$$T_{20}^{-1} \eta_2(\gamma, r) := \eta_2(\gamma, r) - \frac{\beta^2 - \alpha^2}{\gamma^2 - \beta^2} \frac{W[\eta_2(\gamma, r); \xi_2(\beta, r)]}{W[\xi_2(\beta, r); \zeta_2(\alpha, r)]} \zeta_2(\alpha, r) = \eta_0(\gamma, r). \tag{17}$$

Using the definitions of $\eta_2(\gamma, r)$, $\zeta_2(\alpha, r)$ and $\xi_2(\beta, r)$ this expression can easily be verified. It is straightforward to show from (1) that the operator T_{20} satisfies the intertwining relation

$$D_2(r)T_{20}=T_{20}D_0(r) \tag{18}$$

where D_0 and D_2 denote the differential operators

$$D_i(r):=\frac{1}{h(r)}\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\lambda_0^2-\frac{1}{4}}{r^2}+k_0^2-V_i(r)\right] \qquad i=0,2. \tag{19}$$

The existence of the inverse operator (17) together with the validity of the intertwining relation (18) are the characterizing features of *transformation operators*, which have been thoroughly studied by Levitan [20], who has also shown their use in inverse problems. Consequently, the operator T_{N0} corresponding to the generalized Darboux transformation (8) is also a transformation operator because (8) corresponds to an iterated form of the transformation (7).

3. Darboux transformations and integral equations

Inverse scattering theories are usually formulated in terms of integral equations [1, 2]. Prominent examples are the Marchenko and the Gel'fand–Levitan integral equations for problems at fixed angular momentum. For fixed energy an analogous integral equation has been derived by Burdet *et al* [19] and also for the generalized inverse scattering problem (1) an integral equation can be given [14]. In the following we consider the relationship between the Darboux transformations (8) and the corresponding integral equations.

We construct the integral equations starting from the Darboux transformations (8). The key for this procedure is the relation

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{W[\eta_0(\gamma,r);\xi_0(\beta_i,r)]}{\gamma^2-\beta_i^2}=-h(r)\eta_0(\gamma,r)\xi_0(\beta_i,r) \tag{20}$$

which is easily obtained from (1). Integrating (20) from a constant value c to r yields

$$\frac{W[\eta_0(\gamma,r);\xi_0(\beta_i,r)]}{\gamma^2-\beta_i^2}=\frac{W[\eta_0(\gamma,s);\xi_0(\beta_i,s)]^{s=c}}{\gamma^2-\beta_i^2}-\int_c^r \mathrm{d}s\, h(s)\eta_0(\gamma,s)\xi_0(\beta_i,s). \tag{21}$$

In order to derive the standard integral equations of inverse scattering theory the boundary conditions

$$W[\eta_0(\gamma,r);\xi_0(\beta_i,r)]^{r=c}=0 \qquad i=1,2,\ldots,N \tag{22}$$

are required. Thus the Darboux transformations (8) can be cast into the form

$$\eta_2(\gamma,r)=\eta_0(\gamma,r)+\int_c^r \mathrm{d}s\, h(s)K(r,s)\eta_0(\gamma,s) \tag{23}$$

where

$$K(r,s)=\xi_2^T(r)\xi_0(s). \tag{24}$$

The relation (23) is the integral expression satisfied by a generalized transformation operator [2, 20] $K(r,s)$, appearing in the Gel'fand–Levitan procedure ($h(r)=1$, $c=0$)

and in the integral equation of Burdet *et al* [19] ($h(r)=1/r^2$, $c=0$). The symmetry property of Darboux transformations (10–12, 18) leads directly to the characteristic symmetry property of the kernel $K(r,s)$

$$D_2(r)K(r,s)=D_0(s)K(r,s). \tag{25}$$

Summarizing all these features of $K(r,s)$ we can conclude that K is a generalized transformation operator. Hence, K satisfies the integral equation

$$K(r,t)=Q(r,t)-\int_c^r \mathrm{d}s\, h(s)K(r,s)Q(s,t) \qquad \begin{matrix}\text{case 1: } c\leqslant t\leqslant r\\ \text{case 2: } c\geqslant t\geqslant r\end{matrix} \tag{26}$$

where $Q(r,t)$ is a symmetric kernel obeying the relation

$$D_0(r)Q(r,t)=D_0(t)Q(r,t). \tag{27}$$

In our case the symmetric kernel $Q(r,t)$ is defined by

$$Q(r,t:=\begin{cases}\displaystyle\sum_{i=1}^{N} C_i\eta_0(\alpha_i,r)\zeta_0(\alpha_i,t) & \begin{matrix}\text{case 1: } c\leqslant r\leqslant t\\ \text{case 2: } c\geqslant r\geqslant t\end{matrix}\\ \displaystyle\sum_{i=1}^{N} C_i\eta_0(\alpha_i,t)\zeta_0(\alpha_i,r) & \begin{matrix}\text{case 1: } c\leqslant t\leqslant r\\ \text{case 2: } c\geqslant t\geqslant r\end{matrix}\end{cases} \tag{28}$$

and yields via the integral equation (26) the specific transformation kernel (24).

Subject to the boundary condition (22) the integral equation is valid for arbitrary solutions $\eta_0(\alpha_i,r)$, $\zeta_0(\alpha_i,r)$, $\xi_0(\beta_i,r)$ of the Sturm–Liouville equation (1) with the potential $V_0(r)$. The Darboux transformations generate a wide class of exactly solvable models associated with degenerate kernels in the general integral equation (26). The degenerate Gel'fand–Levitan integral equation emerges as a special case ($h(r)=-1$, $c=0$) of our procedure. In this case the use of regular solutions for $\eta_0(\alpha_i,r)$, $\zeta_0(\alpha_i,r)$ and Jost solutions $\xi_0(\beta_i,r)$ leads directly to the exactly solvable model of Bargmann [16, 18]. For $h(r)=1/r^2$, the class of Darboux transformations contains the solutions of the integral equation of Burdet *et al* [19] with degenerate kernels. Darboux transformations with other choices of the wavefunctions $\zeta_0(\alpha_i,r)$, $\xi_0(\beta_i,r)$ and different $h(r)$ lead to new exactly solvable models which are the subject of the next section.

4. Exactly solvable models

The Darboux transformations (8) provide us with exact analytic expressions for the solutions of the Sturm–Liouville equation (1). These wavefunctions are given in terms of the solutions to an *a priori* known reference potential. Using scattering theory we can derive expressions for the S-matrix and the corresponding potential analytically. These so-called exactly solvable models depend on N parameter pairs (α_i,β_i), $i=1,2,\ldots,N$ and the chosen reference problem. In many cases the S-matrix reduces to a very simple expression which can easily be applied to analyse realistic scattering data [13].

In the past exactly solvable models have been formulated and applied mainly for scattering problems at fixed energy [12] and at fixed angular momentum [16, 18]. The study of the general Sturm–Liouville equation (1) allows us not only to consider both standard scattering problems at the same time but also to investigate more complex situations where E and λ^2 are mutually linear dependent [14]. We consider the cases associated with the function

$$h(r)=\frac{a+br^2}{r^2}+h_\gamma(r), \tag{29}$$

where a, b are arbitrary constants and $h_\gamma(r)$ is the form factor of a γ^2-dependent potential term which vanishes asymptotically and fulfills the conditions for potentials in quantum scattering theory. With (29) the Sturm–Liouville equation (1) can be reduced to a Schrödinger equation with k and λ depending on the continuous parameter $\theta=\gamma$,

$$k^2(\gamma)=k_0^2-b\gamma^2 \qquad \lambda^2(\gamma)=\lambda_0^2+a\gamma^2. \tag{30}$$

The inclusion of $h_\gamma(r)$ is a slight generalization compared to the cases considered by Rudyak and Zakhariev [14]. Assuming $h_\gamma(r)\equiv 0$ and fixing $a=0$ or $b=0$ reduces the general scattering problem to a standard one at fixed E or at fixed l, respectively. For $a=0$ and $b=0$, $h_\gamma(r)\not\equiv 0$ we obtain the cases studied by Chadan and Musette [21].

In the following we give some examples of exactly solvable models generated by the Darboux transformations (8) of the general Sturm–Liouville equation (1). In particular, we study the case $a\neq 0$, $b\neq 0$ which is new and has not been considered previously. We restrict ourselves to the three main cases, where $\xi_0(\beta_i, r)$ and $\zeta_0(\alpha_i, r)$ are chosen to be regular, irregular or Jost solutions.

(1) We assume $\xi_0(\beta_i, r)$ and $\zeta_0(\alpha_i, r)$ to be

$$\xi_0(\beta_i, r):=f_0(-k(\beta_i), \lambda(\beta_i), r) \qquad \zeta_0(\alpha_i, r)=f_0(k(\alpha_i), \lambda(\alpha_i), r) \tag{31}$$

where $f_0(k, \lambda, r)$ denotes Jost solutions characterized by their asymptotic behaviour,

$$f_0(k, \lambda, r)\underset{r\to\infty}{\to} \mathrm{e}^{ikr}. \tag{32}$$

Performing the Darboux transformation we determine the corresponding S-matrix $S(k(\gamma), \lambda(\gamma))$ from the asymptotic behaviour of the regular solutions. For N pairs (α_i, β_i) we obtain

$$S(k(\gamma), \lambda(\gamma))=S_0(k(\gamma), \lambda(\gamma))\prod_{i=1}^{N}\frac{(k(\gamma)-k(\alpha_i))}{(k(\gamma)+k(\alpha_i))}\frac{(k(\gamma)+k(\beta_i))}{(k(\gamma)-k(\beta_i))}. \tag{33}$$

The function $S_0(k(\gamma), \lambda(\gamma))$ denotes the S-matrix corresponding to the reference potential $V_0(r)$. In this case the restriction $a\neq 0$, $b\neq 0$ is not essential for the S-matrix (33) because only Jost solutions with their simple asymptotic behaviour (32) occur in the derivation. Using the relationships (30) the S-matrix (33) can also be cast into the form

$$S(k(\gamma), \lambda(\gamma))=S_0(k(\gamma), \lambda(\gamma))\prod_{i=1}^{N}\left(\frac{(k(\gamma)+k(\beta_i))}{(k(\gamma)+k(\alpha_i))}\right)^2\frac{\gamma^2-\alpha_i^2}{\gamma^2-\beta_i^2} \tag{33a}$$

which is a straightforward generalization of the S-matrix obtained in the rational scheme of Lipperheide and Fiedeldey [12]. Furthermore, it should be remarked that there are no additional restrictions on the values $k(\alpha_i), k(\beta_i), i=1, 2, \ldots, N$. Choosing $\beta_i=\alpha_i^*$, $i=1, 2, \ldots, N$ leads to a unitary S-matrix (assuming γ, λ_0, k_0, a, b to be real) and a real potential.

(2) We consider the choice

$$\xi_0(\beta_i, r) := \varphi_0(k(\beta_i), \lambda(\beta_i), r) \qquad \zeta_0(\alpha_i, r) := f_0(-k(\alpha_i), \lambda(\alpha_i), r) \tag{34}$$

where $\varphi_0(k(\beta_i), \lambda(\beta_i), r)$ is the regular solution characterized by its behaviour at the origin

$$\varphi_0(k(\beta_i), \lambda(\beta_i), r) \underset{r\to\infty}{\to} r^{\lambda+1/2}. \tag{35}$$

Applying the Darboux transformation and evaluating the asymptotic behaviour of the corresponding regular solution we can define an S-matrix only under the restriction $\operatorname{Im} k(\beta_i) \neq 0$. We obtain

$$S(k(\gamma), \lambda(\gamma)) = S_0(k(\gamma), \lambda(\gamma)) \prod_{i=1}^{N} \frac{(k(\gamma)+k(\alpha_i))}{(k(\gamma)-k(\alpha_i))} \frac{(k(\gamma)+k(\beta_i))}{(k(\gamma)-k(\beta_i))} \tag{36}$$

where the sign in front of $k(\beta_i)$ is as shown for $\operatorname{Im} k(\beta_i) > 0$, and must be reversed for $\operatorname{Im} k(\beta_i) < 0$. In both regions the restriction $b \neq 0$ is essential for the determination of the S-matrix (36) because for the cases at fixed k another limit has to be taken. In the limit $a = 0$ the S-matrix (36) corresponds to the result of Theis [16] and consequently the restriction $a \neq 0$ is not essential. For real $k(\beta_i)$ an S-matrix can only be defined in the fixed energy scattering problem. This singular case has been outlined in detail elsewhere [17] and is not a simple limit of (36).

(3) We assume

$$\xi_0(\beta_i, r) := \varphi_0(k(\beta_i), \lambda(\beta_i), r) \qquad \zeta_0(\alpha_i, r) := \varphi_0(k(\alpha_i), \lambda(\alpha_i), r). \tag{37}$$

Here we obtain

$$S(k(\gamma), \lambda(\gamma)) = S_0(k(\gamma), \lambda(\gamma)) \prod_{i=1}^{N} \frac{(k(\gamma)+k(\alpha_i))}{(k(\gamma)-k(\alpha_i))} \frac{(k(\gamma)+k(\beta_i))}{(k(\gamma)-k(\beta_i))}. \tag{38}$$

Again the sign in front of $k(\alpha_i)$ is as shown for $\operatorname{Im} k(\alpha_i) > 0$, and must be reversed for $\operatorname{Im} k(\alpha_i) < 0$, and similarly for $k(\beta_i)$. As in case (2) an S-matrix can be given for real $k(\alpha_i)$, $k(\beta_i)$ only at fixed energy. This is the so-called non-rational scheme [12] which, however, is not the limit of (38) for $b = 0$.

The behaviour of the associated potentials near the origin and at asymptotic distances follows from that of different solutions $\xi_0(\beta_i, r)$, $\zeta_0(\alpha_i, r)$. Without going into the details we obtain

$$V_2(r) - V_0(r) \underset{r\to\infty}{\to} C \frac{\mathrm{d}h}{\mathrm{d}r}, \tag{39}$$

$$V_2(r) - V_0(r) \underset{r\to 0}{\to} D \tag{40}$$

where C and D are constants depending on the specific values $k(\alpha_i)$, $k(\beta_i)$.

5. Conclusions

We have studied Darboux transformations of a Sturm–Liouville equation which allows not only a unified description of the inverse scattering problems at fixed energy and at fixed angular momentum but also the treatment of more general inverse scattering problems including mutual linear dependence of λ^2 and k^2. Of particular interest are twice-iterated Darboux transformations which are the underlying transformations of the rational and non-rational schemes [12, 17] and of the solutions of Theis [16]. Starting from these twice-iterated Darboux transformations we have formulated a matrix generalization leading to very compact expressions suitable for application to realistic scattering systems.

Studying the features of the Darboux transformations we have established a symmetry property which is of great importance to understand the connection with the relations of inverse scattering theory. It turns out that the Darboux transformations are realizations of generalized transformation operators as studied by Levitan [20].

The symmetry property is important to reveal the relationship between Darboux transformations and the integral equations of inverse scattering theory. Starting from the Darboux transformations and their properties we can derive generalized integral equations formally equivalent to those of inverse scattering theory [2]. In analogy to the Bargmann potentials, the Darboux transformations always correspond to solutions of the integral equations with degenerate symmetric kernel.

In principle the symmetric kernel contains the spectral information of the scattering system. However, there are no simple relationships with the Jost functions or the S-matrix in the general case because the reference solutions $\xi_0(\beta_i, r)$, $\zeta(\alpha_i, r), i = 1, 2, N$ are arbitrary. It is obvious that this will be a very difficult task which can be done only in specific cases.

These Darboux transformations enable us to define a wide and unified class of exactly solvable models. We have studied some new schemes giving their S-matrices explicitly. These extended scattering schemes are of particular interest for studying problems where λ^2- or k^2-dependent potentials are involved. Applications of this type of models are in progress.

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